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Note

Stabilization of Cowell's Classical Finite Difference Method for Numerical Integration

1. Introduction

The classical finite difference methods [1] used for the numerical integration of ordinary differential equations integrate polynomials exactly, i.e., without truncation errors. Numerical instabilities occur when these classical methods are applied to ordinary differential equations whose solutions are not polynomials. Only a few of the many methods available for numerical integration take advantage of properties of the solution if these are known in advance. Such integration methods have been developed previously by Brock and Murray [2] and Dennis [3] for exponential solutions by Urabe and Mise [4], Gautschi [5] and Salzer [6] for periodic or oscillatory solutions, by Stiefel and Bettis [7–10], and by Sheffield [11] for the numerical integration of products of Fourier and ordinary polynomials. In [12] the author illustrated the stability of the Stiefel–Bettis method for oscillating phenomena in weakly nonlinear mechanical systems with two degrees of freedom described by coupled Duffing equations.

All previous work has been concerned with a special type of solutions. Therefore the purpose of this study is to develop a generalized modification of Cowell's classical finite difference method that will stabilize the numerical integration of a system that can be considered as a small perturbation of an auxiliary system for which an exact solution is known. The highest coefficient of Cowell's classical integration formula will be altered by requiring this formula to integrate the exact solution to the auxiliary system. Hence the instability that is inherent in the Cowell method will be reduced. Although the basic idea of this modified method has been introduced previously, the suggestion to modify only the highest coefficient of Cowell's formula is new. The broad applicability of this modified integration method to perturbation theory is obvious. The stabilizing effect is illustrated by the example of the Duffing equation forced by a harmonic function. As is well known this equation describes many important oscillating phenomena in nonlinear physical systems.

2. The Modified Integration Method

For a differential equation of the second order in which the first derivatives are absent,

$$\ddot{x} = f(x, t), \tag{2.1}$$

the double integration of a given function $f(x, t)$, tabulated at equally spaced values of the independent variable t, with step length h, can be performed by the use of Cowell's classical integration formula [1]

$$\begin{aligned}\Delta^2 x(n) = h^2[f_n &+ \beta_1 \Delta^1 f(n-0.5) + \beta_2 \Delta^2 f(n) + \beta_3 \Delta^3 f(n-0.5) + \beta_4 \Delta^4 f(n-1) \\ &+ \beta_5 \Delta^5 f(n-1.5) + \beta_6 \Delta^6 f(n-2) + \cdots + \beta_k \Delta^k f(n+1-(k/2))],\end{aligned} \tag{2.2}$$

with the definitions

$$\begin{aligned} t_n = nh, \qquad x_n = x(t_n), \qquad f_n &= f(x_n, t_n), \qquad n \text{ integer}, \\ \Delta^0 f(n) &= f_n, \\ \Delta^1 f(n-0.5) &= \Delta^0 f(n) - \Delta^0 f(n-1), \\ \Delta^2 f(n) &= \Delta^1 f(n+0.5) - \Delta^1 f(n-0.5), \\ \ldots . \end{aligned} \tag{2.3}$$

A dot means differentiation with respect to t.

For Cowell's classical method the coefficients β are obtained from the assumption that the formula (2.2) integrates polynomials exactly. This yields the following coefficients which are independent of the interpolation nodes.

$$\begin{aligned}\beta_1 = 0, \qquad \beta_2 = 1/12, \qquad \beta_3 = 0, \qquad \beta_4 = -1/240, \qquad \beta_5 = -1/240, \\ \beta_6 = -221/60480, \ldots .\end{aligned} \tag{2.4}$$

By using the formula (2.2) with a finite set of coefficients β for the integration of equations whose solutions are not polynomials, numerical instabilities occur due to the truncation errors. In our modified integration method Cowell's classical integration formula will be retained, but the coefficients β will be altered in order to reduce these instabilities. This effect will be called stabilization by modification of the integration coefficients. Let $y(t)$ be the exact solution to the equation

$$\ddot{x} = f^{(0)}(x, t) \tag{2.5}$$

which we assume to be slightly different from the equation (2.1). By a slightly different equation we mean that $f^{(0)}(x, t)$ has nearly the same behavior as $f(x, t)$ such that (2.1) may be considered as a small perturbation of (2.5).

We propose to modify only the highest coefficient of Cowell's integration formula (2.2), the other coefficients retain their classical Cowell values. Hence it is easily seen that the modified method of order k will perform exactly the numerical integration of polynomials up to the degree integrated exactly by Cowell's classical method of order $k - 1$. In order to reduce the numerical instabilities of the integration of (2.1), we require the formula (2.2) to perform exactly the integration of the exact solution $y(t)$ to (2.5). Hence inserting

$$\begin{aligned} x(t) &\equiv y(t), \\ f(x, t) &\equiv f^{(0)}(y(t), t) \equiv g(t), \end{aligned} \tag{2.6}$$

into (2.2), we obtain for the highest coefficient in our modified method

$$\begin{aligned} \beta_k = [&\Delta^2 y(n)/h^2 - g_n - \beta_1 \Delta^1 g(n - 0.5) - \beta_2 \Delta^2 g(n) - \cdots \\ &- \beta_{k-1} \Delta^{k-1} g(n + (3/2) - (k/2))]/\Delta^k g(n + 1 - (k/2)). \end{aligned} \tag{2.7}$$

The modified coefficient depends on the characteristics of the considered equation, on the integration step length h and on the interpolation nodes indicated by n.

Using the definitions of the differences (2.3) the integration formula (2.2) can be written as follows.

$$\begin{aligned} x_{n+1} = 2x_n - x_{n-1} + h^2(&\gamma_1 f_{n+1} + \gamma_0 f_n + \gamma_{-1} f_{n-1} + \gamma_{-2} f_{n-2} + \gamma_{-3} f_{n-3} \\ &+ \gamma_{-4} f_{n-4} + \gamma_{-5} f_{n-5} + \cdots), \end{aligned} \tag{2.8}$$

where the coefficients γ are linear expressions in the coefficients β [12].

If $\Phi(x, t)$ represents the nonlinear part of $f(x, t)$ with respect to x such that

$$f(x, t) = -\omega^2 x + \Phi(x, t), \tag{2.9}$$

then by substituting f_{n+1} into (2.8), we have the following formula.

$$x_{n+1} = [2x_n - x_{n-1} + h^2(\gamma_1 \Phi_{n+1} + \gamma_0 f_n + \gamma_{-1} f_{n-1} + \cdots)]/(1 + \gamma_1 \omega^2 h^2), \tag{2.10}$$

with

$$\Phi_{n+1} = \Phi(x_{n+1}, t_{n+1}).$$

Now x_{n+1} is calculated using some predictor formula and the corresponding value is substituted into the right-hand side of the formula (2.10). Finally this formula is iterated and used as corrector formula.

Remark 1. Although it has been formulated only for a single differential equation, the method can be easily extended to systems of equations by introducing vector functions for $x, f, \beta_i, y, g, \gamma_i, \Phi$. For further details of this self-evident generalization, we refer to [12] where a similar procedure has been applied to a system with two degrees of freedom.

Remark 2. For a particular system an efficient reduction of the form of the modified coefficient can often be developed to streamline its computation. This is especially the case if the unperturbed system (2.5) describes the motion of the harmonic oscillator (cf. [5–11]).

Remark 3. Instead of computing the highest coefficient β at every step of integration, we could try to calculate this coefficient just once (e.g. at the interpolation node corresponding with $n = 0$), thus reducing the amount of computing. The coefficient obtained in this way will be called the constant modified coefficient to distinguish it from the variable modified coefficient changing at every interpolation node. The numerical example of Section 3 shows that recomputation of the highest coefficient β at every step is necessary, otherwise no appreciable improvement of the accuracy of the integration is obtained compared to Cowell's classical integration. The idea of recomputation at every step has been introduced previously by Gautschi [5].

3. Numerical Example

Since Cowell's classical method becomes more stable with increasing order, stabilization by modification of the integration coefficients is more efficient for lower orders. Therefore we shall now illustrate the stabilizing effect of the modified integration method of order 2.

Let us consider the Duffing equation forced by a harmonic function

$$\ddot{x} + \omega^2 x + \epsilon x^3 = F \cos \Omega t, \tag{3.1}$$

with the following values of the parameters.

$$\omega = 1, \qquad \epsilon = 1, \qquad F = 0.002, \qquad \Omega = 1.01. \tag{3.2}$$

It is to be noted that the nonlinear term in this equation has a coefficient which is not small.

By Urabe's method applied to Galerkin's procedure [13, 14] we computed the Galerkin approximation of order 9 to a periodic solution having the same period as the forcing term, with a precision of the coefficients of 10^{-12}.

$$\begin{aligned} x_G(t) = {} & 0.200179477536 \cos \Omega t + 0.000246946143 \cos 3\Omega t \\ & + 0.000000304014 \cos 5\Omega t + 0.000000000374 \cos 7\Omega t \\ & + 0.000000000000 \cos 9\Omega t. \end{aligned} \tag{3.3}$$

Under certain assumptions the forced Duffing equation (3.1) can be considered as the perturbed equation of the Duffing equation forced by a Jacobian elliptic function

$$\ddot{x} + \omega^2 x + \epsilon x^3 = F\, cn(\Omega_0 t \mid m), \tag{3.4}$$

since the Jacobian elliptic function $cn(u \mid m)$ reduces to the circular cosine function $\cos u$ when the parameter m of the Jacobian elliptic function tends to zero [15]. Therefore m has to be small and Ω_0 nearly equal to Ω. Hence for the numerical integration of the perturbed equation (3.1) we propose to modify the highest integration coefficient β by requiring Cowell's classical formula to integrate the exact solution of the considered unperturbed equation (3.4).

The exact solution to the elliptic forced Duffing equation (3.4), satisfying the initial conditions

$$y(0) = A \qquad \text{and} \qquad \dot{y}(0) = 0, \tag{3.5}$$

is the Jacobian elliptic function

$$y(t) = A\, cn(\Omega_0 t \mid m) \tag{3.6}$$

if the following relations between the parameters hold [16];

$$\Omega_0 = (\omega^2 - (F/A) + \epsilon A^2)^{1/2} \qquad \text{and} \qquad m = \epsilon A^2/2\Omega_0{}^2. \tag{3.7}$$

Taking for A the value of the Galerkin approximation x_G at $t = 0$, the values of Ω_0 and m are found to be

$$\Omega_0 = 1.014983824649 \qquad \text{and} \qquad m = 0.019496786481. \tag{3.8}$$

Therefore the exact solution of the unperturbed equation is completely known and we can compute the highest coefficient β from (2.7), where

$$g(t) = [(F/A) - \omega^2 - \epsilon A^2\, cn^2(\Omega_0 t \mid m)]A\, cn(\Omega_0 t \mid m). \tag{3.9}$$

Then we perform the numerical integration of the perturbed equation using for β_2 Cowell's classical coefficient, the constant modified coefficient and the variable modified coefficient, respectively. The values of x and f in the previous points t_0, t_{-1} are computed from the Galerkin approximation. Table I shows the Galerkin approximation and the results of the integrations with step length $h = 2\pi/45\Omega$. Table II gives the absolute values of the maximum differences between the results of the integrations and the Galerkin approximation for various values of the step length. Compared to the Cowell integration, the accuracy is only improved if the variable modified coefficient is used. This reduces the numerical instabilities of Cowell's classical method, revealing an average improvement

TABLE I

Numerical integration of the forced Duffing equation with $h = 2\pi/45\Omega$

n	Galerkin approximation	Cowell integration	Modified integration β_2 constant	Modified integration β_2 variable
50	0.153222617706	0.153221820532	0.153223122231	0.153222617752
100	0.034637560945	0.034635336024	0.034639160231	0.034637561153
150	−0.099842944818	−0.099845665393	−0.099840846831	−0.099842944686
200	−0.188230597866	−0.188231801036	−0.188229677172	−0.188230598359
250	−0.188230597866	−0.188228299239	−0.188232280437	−0.188230598913
300	−0.099842944819	−0.099837091488	−0.099847220777	−0.099842945545
350	0.034637560945	0.034644488045	0.034632356420	0.034637561485
400	0.153222617706	0.153226926360	0.153219326288	0.153222620013
450	0.200426728066	0.200425293937	0.200427793954	0.200426731316
500	0.153222617706	0.153215037421	0.153228206330	0.153222619452
550	0.034637560945	0.034627120554	0.034645328565	0.034637558931
600	−0.099842944818	−0.099851047392	−0.099836796644	−0.099842950874
650	−0.188230597866	−0.188231841141	−0.188229634560	−0.188230605602
700	−0.188230597866	−0.188223555630	−0.188235835473	−0.188230602390
750	−0.099842944819	−0.099830760861	−0.099851979205	−0.099842941518
800	0.034637560945	0.034648775037	0.034629122124	0.034637572665
850	0.153222617706	0.153227118404	0.153219164611	0.153222633338
900	0.200426728066	0.200421900915	0.200430333003	0.200426739068
950	0.153222617706	0.153210744788	0.153231436465	0.153222615493
1000	0.034637560945	0.034624856197	0.034647049050	0.034637543126

TABLE II

Error vs step length

Step length	Cowell integration	Modified integration β_2 constant	Modified integration β_2 variable
$2\pi/45\Omega$	1.4×10^{-5}	1.0×10^{-5}	1.8×10^{-8}
$2\pi/89\Omega$	8.4×10^{-7}	6.4×10^{-7}	1.5×10^{-9}
$2\pi/179\Omega$	5.1×10^{-8}	3.9×10^{-8}	1.3×10^{-10}

of about three decimal digits. Further, the modified integration using this coefficient and performed with step length $2\pi/45\Omega$ yields slightly better results than the Cowell integration with step length $2\pi/179\Omega$, the total computation time being reduced by a factor 2. Therefore it is worthwhile to apply the modified algorithm since it enables us to use increased step sizes and to decrease the total amount of computing.

All computations have been carried out on the computer CDC 6400 at the University of Brussels.

References

1. P. Henrici, "Discrete Variable Methods in Ordinary Differential Equations," Wiley, New York, 1962.
2. P. Brock and F. J. Murray, The use of exponential sums in step-by-step integration, *Math. Tables Aids Comput.* **6** (1952), 63–78.
3. S. C. R. Dennis, The numerical integration of ordinary differential equations possessing exponential type solutions, *Proc. Cambridge Philos. Soc.* **56** (1960), 240–246.
4. M. Urabe and S. Mise, A method of numerical integration of analytic differential equations, *J. Sci. Hiroshima Univ., Ser. A-I Math*, **19** (1955), 307–320.
5. W. Gautschi, Numerical integration of ordinary differential equations based on trigonometric polynomials, *Numer. Math.* **3** (1961), 381–397.
6. H. E. Salzer, Trigonometric interpolation and predictor–corrector formulas for numerical integration, *Z. Angew. Math. Mech.* **42** (1962), 403–412.
7. E. Stiefel and D. G. Bettis, Stabilization of Cowell's method, *Numer. Math.* **13** (1969), 154–175.
8. D. G. Bettis, Numerical integration of products of Fourier and ordinary polynomials, *Numer. Math.* **14** (1970), 421–434.
9. D. G. Bettis, Stabilization of finite difference methods of numerical integration, *Cel. Mech.* **2** (1970), 282–295.
10. E. Stiefel and G. Scheifele, "Linear and Regular Celestial Mechanics," Springer–Verlag, Berlin, 1971.
11. C. Sheffield, Generalized multistep methods with an application to orbit computation, *Cel. Mech.* **1** (1969), 46–58.
12. R. Van Dooren, Numerical computation of forced oscillations in coupled Duffing equations, *Numer. Math.* **20** (1973), 300–311.
13. M. Urabe, Galerkin's procedure for nonlinear periodic systems, *Arch. Rational Mech. Anal.* **20** (1965), 120–152.
14. M. Urabe and A. Reiter, Numerical computation of nonlinear forced oscillations by Galerkin's procedure, *J. Math. Anal. Appl.* **14** (1966), 107–140.
15. M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," Dover, New York, 1965.
16. C. S. Hsu, On the application of elliptic functions in nonlinear forced oscillations, *Quart. Appl. Math.* **17** (1959), 393–407.

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